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Using of Method of Moments for Solve a Class of Optimal Control Problem

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Abstract

In this paper, we solve a class of optimal control problem where the objective function given by the ratio of two integrals. We propose an alternative method for computing effectively the solution of fixed-terminal-time, fractional optimal control problems when they are given in non-linear forms. This method works well when the nonlinearities in the control variable can be expressed as polynomials. The essential of this proposal is the transformation of a non-linear, non-convex optimal control problem into an equivalent optimal control problem with linear and convex structure. The method is based on global optimization of polynomials by the method of moments. With this method we can determine either the existence or lacking of minimizers. In addition, we can calculate generalized solutions when the original problem lacks of minimizers. We also present the numerical schemes to solve some examples.

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INTRODUCTION

Optimal control problem and its various branches has been much attention in long time. Fractional optimal control problem as a form of such problems in various classes has been studied. Fractional optimal control problems in a particular form where the objective functional is given by the ratio of two integrals, is considered by Stancu-Minasian [1], Bhatt [2], in a more general framework, by Miele [3] and on problems with affine integrands and linear dynamics with respect to state and control by Bykadorov et.al [4]. Despite the simplicity of these problems, the standard optimal control theory cannot directly be used to solve them. Stancu-Minasian [1] suggested to face the general fractional optimal control problem applying the Dinkelbachs method [5, 6], which is used in fractional programming to remove the denominator in the objective function.

Recently Meziat et al. [7] presented the method of moments where it is based on moment problem for solving the optimal control problem. Moment problem is a very common problem in physics and engineering. The characterization of sequences that are moments of some measure is a basic problem in the theory of moments. Moment problems occur frequently in spectral estimation and in particular in speech processing, geophysics, sonar and radar, and many other areas [8,9]. A one survey of wide range of approaches to the moment problem and its application is in [10]. There are various approaches in estimating the measure when we have a finite number of moments. In this paper, the idea is to approximate the measure by Dirac measure. When we use the method of moments, the control variable need not be in the linear form, because by using of moments the problem is transformed to in linear form in term of control variable in space of moments. Is the considered problem has minimizer or not, is the important corollary of this method.

The paper is organized as follows: Section 2 introduced fractional optimal control problems in a particular form where the objective functional is given by the ratio of two integrals. Some basic definition of measures and moments appear in section 3. In section 4, the modified problem is discussed. The computational estimation of the solution by Computational treatment title present in section 5. The conclusion remark appears in sections 6.

MATERIAL AND METHODS

Problem Formulation

A general form of the optimal control where the objective functional is given by the ratio of two integrals, as bellow:

$$J = \min imize \frac{\int_a^b f(t,x(t),u(t))dt}{\int_a^b g(t,x(t),u(t))dt} \qquad (1)$$

$$x'(t) = h(t,x(t),u(t))$$

$$x(a) = x_a$$

where the functions f,g and h can be represented as polynomials in the control variable

$$f(t,x(t),u(t)) = \sum_{i=1}^{n_1} a_i(t,x(t))u^i(t),$$

$$g(t,x(t),u(t)) = \sum_{i=1}^{n_1} b_i(t,x(t))u^i(t),$$

$$h(t,x(t),u(t)) = \sum_{i=1}^{n_1} c_i(t,x(t))u^i(t).$$

And we suppose that

$$\forall t \in [a,b] \quad \int_a^b g(t,x(t),u(t)) > 0$$

RESULTS

Measures and Moments

In this section we present some definitions of measures and moments of [11-12] which use to method of moments. Let X be a subset of R^n and $\mathrm{B}(X)$ denotes the Borel $\sigma-$algebra.

Definition1: A signed measure is a function $\mu:\mathrm{B}(X)\to R\cup\infty$ such that $\mu(\phi)=0$ and $\mu(\cup A_i) = \sum_{i\in N} \mu(A_i)$ for any pairwise disjoint $A_i \in \mathrm{B}(X)$.

Definition 2: A positive measure is a signed measure which takes only none negative values.

Definition 3: A probability measure μ on X is a positive measure such that $\mu(X)=1$.

Definition 4: A Dirac measure at $x=\zeta$ denoted by $\delta_{x=\zeta}$ is a probability measure such that

$$\delta(A) = \begin{cases} 1 & \zeta \in A \\ 0 & \zeta \notin A \end{cases}$$

Remark 1: If $M(X)$ denotes the Banach space of signed measures supported on X, then a measure $\mu \in M(X)$ can be interpreted as a function that takes any subset of X and returns a real number.

Definition 5: Given a real vector $x\in R^n$ and an integer vector $\alpha \in N^n$, a monomial is defined as $x^\alpha = x_1^{\alpha_1}.x_2^{\alpha_2}....x_n^{\alpha_n}$ and the degree of the monomial is equal to $|\alpha| = \sum_{i=1}^n \alpha_i$.

Definition 6: Given a measure $\mu \in \mu(X)$, the real number

$$y_\alpha = \int_X x^\alpha \mu(dx) \qquad (2)$$

is called its moment of order $\alpha \in N^n$.

For example if $x = (x_1,x_2) \in X \subset R^2$, second order moments are

$$y_{00} = \int_X \mu(dx), \quad y_{10} = \int_X x_1\mu(dx),$$

$$y_{01} = \int_X x_2\mu(dx), \quad y_{20} = \int_X x_1^2\mu(dx),$$

$$y_{11} = \int_X x_1x_2\mu(dx), \quad y_{02} = \int_X x_2^2\mu(dx)$$

The sequence $(y_\alpha)_\alpha \in N^\alpha$ is called the sequence of moments of the measure μ and given $q\in N$ the truncated sequence $(y_\alpha)_{|\alpha|\le q}$ is the vector of moments of degree *q*.

Remark 2: If y is the sequence of moments of a measure μ i.e. if identity (2) hold for all $\alpha \in N^n$, we say that μ is a representing measure for y. A basic problem in the theory of moments concerns the characterization of sequences that are moments of some measure. Indeed a measure on a compact set is uniquely determined by the sequence of its moments.

Let $P_n \in R[X]$ denote the polynomials of degree at most n, for given a sequence $y = (y_\alpha)_{\alpha\in N^n}$ the Riesz linear functional defined as bellow:

$$L_y : R[X] \to R$$

Where for $P_n = \sum_\alpha p_\alpha x^\alpha \in R[X]$ implies that

$$L_y(P_n) = \sum_\alpha p_\alpha y_\alpha.$$

$(p_\alpha)_{|\alpha|\le n}$ is the vector of coefficients of polynomial $P_n(X)$.

In other word the Riesz functional can be interpret as an operator that linearizes polynomials.

Remark 3. If sequence y has a representing measure μ integration of a polynomial $P_n(X)$ w.r.t μ is obtained by applying the Riesz functional L_y on $P_n(X)$ since

$$L_y(P_n) = \sum_\alpha p_\alpha y_\alpha = \sum_\alpha p_\alpha \int x^\alpha \mu(dx) =$$

$$\int \sum_\alpha p_\alpha x^\alpha \mu(dx) = \int P_n(x)\mu(dx)$$

Definition 7: The moment matrix of order n is the matrix $M_q(y)$ such that:

$$l_y(P_n^2(y)) = p_n^t M_q(y) p_n,$$

where p_n is the vector of coefficients of $P_n(X)$. For example if $x=(x_1,x_2)\in R^2$,

$$M_1(y) = \begin{pmatrix} y_{00} & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{pmatrix}$$

Remark 4: The rows and columns of the moment matrix are indexed by vectors $\alpha \in N^n, \beta \in N^n$.

Inspection reveals that indeed the entry (α, β) in the moment matrix is the moment $y_{\alpha+\beta}$. By construction, the moment matrix $M_q(y)$ is symmetric and linear in y.

In special case when $X \subset R$, we denote the moment of order $i \in N$ as bellow

$$m_k = \int_X x^k \mu(dx)$$

Let $m = \{m_k\}$ be the sequence of moments of some probability measure μ_m, with first element $m_0 = 1$ and let $M_q(m)$ be the moment matrix of dimension q, which is composed of all the sequence in R^{q+1} whose entries form a positive semidefinite Hankel matrix [1,5].

$$M_q(m) = \begin{pmatrix} m_0 & m_1 & \ldots. & m_q \\ m_1 & m_2 & \ldots. & m_{q+1} \\ \ldots.. & & \ldots\ldots & \ldots. \\ m_q & m_{q+1} & \ldots.. & m_{2q} \end{pmatrix}$$

The theory of moments identifies those sequence m with

$M_q(m) \succ 0$, that correspond to moments of some probability measure μ_m on R^q.

Modified problem

In this section we use method of Charnes and Cooper and obtain modified problem of (1), for this work we let

$$\gamma = \frac{1}{\int_a^b g(\mathrm{t}, \mathrm{x(t)}, \mathrm{u(t)})}$$

Therefore the problem (1) leads to the following optimal control problem:

$$J = \min \ \gamma \int_a^b f(t, x(t), u(t)) dt$$

$$x'(t) = h(t, x(t), u(t)) \qquad (3)$$

$$\gamma . \int_a^b g(t, x(t), u(t)) dt = 1$$

$$x(a) = x_a$$

With respect to the polynomial form of functions f, g and h, so the Hamiltonian H of the optimal control problem (2) must have a polynomial form in the control variable u:

$$H = H(t, \lambda, \mathrm{x}, \mathrm{u}) = \sum_{i=0}^{N} \alpha_i (t, \lambda, x) u^i, \qquad (4)$$

Where $N = \max\{n_1, n_2, n_3\}$. Thus, the global minimization of H in u:

$$\min_u H(u) = \sum_{i=0}^{N} \alpha_i u^i, \qquad (5)$$

is a problem well suited to be solved by the method of moments [13-14].The essentials of this method follow.

For solving non-convex polynomial programs like (5), we can use the convex hull of the graph of the polynomial H provided it be a coercive function, that is: $\alpha_i > 0$ with even i. We can describe such convex set in the following way:

$$co(graph(H)) = \{\int_R (\mathrm{u}, \mathrm{H(u)}) d\mu(u) : \mu \in \mathrm{P(R)}\} \qquad (6)$$

where $\mathrm{P(R)}$ stands for the family of all probability Borel measures supported in the real line.

Theorem1 [15]: Let $H(u)$ be an even degree, algebraic polynomial whose leader coefficient α_i is positive, then we can express the convex hull of the graph of H as given in (6).

To prove this result, apply the separation theorem of convex analysis. Once we have characterized the convex hull of the graph of *H*, we can obtain the set of all global minima of H by noticing that:

$$\arg\min(\mathrm{H}) \subseteq \mathrm{argmin}(\mathrm{H}_c),$$

where H_c stands for the convex envelope of H. Since H is a coercive polynomial, notice that

$$co(graph(H)) = Epigraph(H_c).$$

Then, we can pose the global optimization problem (6) as the following optimization problem defined in probability measures:

$$\min_{\mu \in P(R)} \int_R H(u) d\mu(u), \qquad (7)$$

whose solution is the family of all probability measures supported in $\arg\min(H)$. See [13].

Theorem 2: [15,16] When H is coercive, the set of solutions of (7) is the set of all probability measures supported in the set of global minima of H, i.e. $\arg\min(H)$.

Corollary 3: [15,16] When $\arg\min(H)$ is the singleton $\{u^*\}$, the Dirac measure $\mu^* = \delta_{u^*}$ is the unique solution of (7).

Now we use the polynomial structure of the objective function H in order to transform the optimization problem (7) into the following optimization problem:

$$\min_{m \in M} \sum_{i=0}^{N} \alpha_i m_i$$

where M is the convex set of all vectors in R^{N+1} whose entries are the algebraic moments of a probability measure supported in the real line. Although this formulation seems very attractive due to the linear form of the objective function and the convex structure of the feasible set, it is still a theoretical formulation not very useful if we do not properly characterize the feasible set composed of moment vectors M. However, this is precisely the question of the classical Truncated Hamburger Moment Problem. Its solution is easily summarized as follows: the closure of M is composed of all vectors in R^{N+1} whose entries form a positive semi definite Hankel matrix [17-18]:

$$\overline{M} = \{(m_i)_{i=0}^{N} \in R^{N+1},\ (m_{i+j})_{i,j=0}^{\frac{N}{2}}$$

$$\textit{is positive semidefinite with } m_0 = 1\}.$$

This result allows us to transform the problem (7) into the mathematical program:

$$\min_{m} \sum_{i=0}^{N} \alpha_i m_i \quad s.t. \quad (m_{i+j})_{i,j=0}^{N/2} \geq 0 \quad m_0 = 1 \qquad (8)$$

which has the form of a semi-definite program. See [20,23] for an introduction to conic and semi-definite programming.

Theorem 4 [16]: When H is a coercive polynomial, the set of solutions of (8) is the set of all vectors $m^* \in R^{N+1}$ whose entries are the algebraic moments of some probability measure supported in $\arg\min(H)$, which is a finite set with N points at the most.

Corollary 5 [16]: When H is a coercive polynomial with a unique global minimum u^*, the program (8) has a unique solution $m^* \in R^{N+1}$ composed by the algebraic moments of the Dirac measure δ_{u^*}.

Thus, $m_i^* = (u^*)^i$, $\forall i = 0,1,2,...,N$.

Hence, the global minimization of the Hamiltonian H can be formulated as:

$$\min_{m} \tilde{H}(t,\lambda,x,u) = \sum_{i=0}^{N} \alpha_i (t,\lambda,x) m_i$$

$$s.t. \quad (m_{i+j})_{i,j=0}^{\frac{N}{2}} \geq 0 \qquad (9)$$

$$m_0 = 1.$$

where the variables t,λ and p are fixed. Notice that any solution $m^*(t,\lambda,x)$ of (9) is composed of the algebraic moments of some probability measure supported in $\arg\min(H(t,\lambda,x))$. Since $\arg\min(H(t,\lambda,x))$ is finite, $m^*(t,\lambda,x)$ can be expressed as:

$$m^* = \sum_{i=1}^{N'} \gamma_i (1, v_i, v_i^2, ..., v_i^N) \qquad (10)$$

Where

$$\arg\min(H(t,\lambda,x)) = \{v_1(t,\lambda,x), v_2(t,\lambda,x), ..., v_{N'}(t,\lambda,x)\}$$

with $N' \leq N$.

Therefore, if $\arg\min(H)$ is the singleton $\{u^*(t,\lambda,x)\}$, the optimal control can be expressed as:

$$u^*(t,\lambda,x) = m_1(t,\lambda,x),$$

because the entries of $m^*(t,\lambda,x)$ are the moments of the Dirac measure $\delta_{u^*(t,\lambda,x)}$. In this work we will solve the non-linear, non-convex problem (1) by working out its convex relaxation:

$$\begin{aligned} &\min_{m(t)} \gamma \int_a^b \sum_{i=0}^{n_1} a_i(t,x) m_i(t)\, dt \\ &x'(t) = \sum_{i=0}^{n_2} b_i(t,x) m_i(t), \\ &\gamma \int_a^b c_i(t,x) m_i(t)\, dt = 1, \qquad (11) \\ &x(a) = x_a, \\ &(m_{i+j=0})_{i,j=0}^{\frac{N}{2}} \geq 0 \quad \text{with } m_0(t) = 1 \ \ \forall t \in (a,b). \end{aligned}$$

Theorem 6: Let us assume that $u^*(t)$ is a minimizer of the optimal control problem (1), then the control vector $m^*(t)$ given as

$$m_i^*(t) = (u^*(t))^i \quad \forall i = 0,1,...,N \qquad (12)$$

is a minimizer of the formulation (11).

Thus, every minimizer of the convex formulation (11) attains the infimum of the non-linear optimal control problem (1).

Computational treatment

Now we focus on the computational estimation of the solution of the formulation (11) as a non-linear mathematical program. We take a discrete net of points $t_0, t_1, t_2, ..., t_l$ on the interval time $[a,b]$, a set of design variables $m(t_r)$ intended to represent the control variables $m \in R^{N+1}$ and the variables $x(t_r)$ intended to represent the state variables $x \in R^n$. When the points t_r are uniformly distributed on the interval $[a,b]$, we obtain the mathematical program:

$$\begin{aligned} &\min_{m,x,\gamma} \sum_{r=0}^{l-1} \int_{rh}^{(r+1)h} \gamma \sum_{i=0}^{n_1} a_i m_i(r)\, dt \\ &\frac{x_r - x_{r-1}}{h} = \sum_{i=0}^{n_2} b_i m_i(r), \quad \forall r = 1,2,...,l \\ &\sum_{r=0}^{l-1} \int_{rh}^{(r+1)h} \gamma c_i m_i(r)\, dt = 1, \qquad (12) \\ &x(a) = x_a, \\ &m_0(r) = 1 \quad \forall r = 0,1,2,...,l \\ &(m_{i+j=0}(r))_{i,j=0}^{\frac{N}{2}} \geq 0 \quad \forall r = 0,1,2,...,l \end{aligned}$$

where $h = \frac{b-a}{l}$ is the uniform distance between the discrete net points. In order to represent the matrix

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inequality condition as a set of non-linear inequalities, we use the fact that all principal sub-determinants of a positive semi-definite matrix are nonnegative [9]. Then, the matrix inequality condition

$$(m_{i+j=0}(r))_{i,j=0}^{\frac{N}{2}} \geq 0 \qquad (13)$$

is expressed as a set of non-linear inequality constraints:

$$D_p(m(r)) \geq 0 \quad \forall p = 1,2,\ldots,s \quad , \quad \forall r = 0,1,\ldots,l$$

where D_p is the explicit form of every principal sub-determinant of the Hankel matrix in (13) and p is the number of its principal sub-determinants. In this way, we have transformed the optimal control problem (1) into a non-linear mathematical program.

Notice that coefficients a_i, b_i and c_i may depend on x and t. In order to solve this kind of high-dimensional, non-linear mathematical programs, we use Gams software.

We explain in full detail some examples of non-linear optimal control problems analyzed by the method of moments proposed here.

Example 1: We illustrate the success of the method proposed in this work by solving the following optimal control problem:

$$J = \min \frac{\int_0^1 (\mathrm{u(t)} - \mathrm{t})^2 dt}{1 + \int_0^1 (\mathrm{x}^2(\mathrm{t}) + \mathrm{u}^2(\mathrm{t}))\,\mathrm{dt}}$$

$$x'(t) = 1 + t - t^2 - x(t) + u^2(t),$$
$$x(0) = 0$$

Solution: Let $\gamma = 1 \Big/ 1 + \int_0^1 (\mathrm{x}^2(\mathrm{t}) + \mathrm{u}^2(\mathrm{t}))\,\mathrm{dt}$,

we obtain the following optimal control problem:

$$J = \min \gamma \int_0^1 (\mathrm{t}^2 - 2t\,u(t) + u^2(t))dt$$
$$x'(t) = 1 + t - t^2 - x(t) + u^2(t),$$
$$\gamma + \gamma \int_0^1 (\mathrm{x}^2(\mathrm{t}) + \mathrm{u}^2(\mathrm{t}))\,\mathrm{dt} = 0$$
$$x(0) = 0$$

Its relaxed formulation (11) takes the form:

$$J = \min \gamma \int_0^1 (\mathrm{t}^2 - 2tm_1(t) + m_2(t))dt$$
$$x'(t) = 1 + t - t^2 - x(t) + m_2(t),$$
$$\gamma + \gamma \int_0^1 (\mathrm{x}^2(\mathrm{t}) + m_2(t))\,\mathrm{dt} = 0,$$
$$\begin{pmatrix} 1 & m_1(t) \\ m_1(t) & m_2(t) \end{pmatrix} \geq 0 \qquad \forall t \in [0,1]$$
$$x(0) = 0.$$

Next we write down this formulation as the discrete, non-linear mathematical program:

$$J = \min \sum_{r=0}^{l-1} \gamma((\mathrm{t}_r^{\ 2} - 2t_r m_1(t_r) + m_2(t_r))$$
$$+ (\mathrm{t}_{r-1}^{\ 2} - 2t_{r-1} m_1(t_{r-1}) + m_2(t_{r-1})))\frac{h}{2}$$
$$\frac{x_r - x_{r-1}}{h} = 1 + t_r - t_r^{\ 2} - x(t_r) + m_2(t_r), \quad \forall\, \mathrm{r} = 1,2,\ldots,\mathrm{l},$$
$$\gamma + \gamma \sum_{r=0}^{l-1} ((\mathrm{x}^2(t_r) + m_2(t_r)) + (\mathrm{x}^2(t_{r-1}) + m_2(t_{r-1})))\frac{h}{2} = 0,$$
$$\begin{pmatrix} 1 & m_1(t_r) \\ m_1(t_r) & m_2(t_r) \end{pmatrix} \geq 0 \qquad \forall r = 0,1,\ldots l,$$
$$x(0) = 0.$$

By choose $h = 0.1$, we obtain the value of objective function as zero and the exact value of objective function is equal 0.

Example 2: In this example we solve the following optimal control problem in similar way of previous example:

$$\min \frac{\int_0^1 (\mathrm{x(t)} - \mathrm{t}^2)^2 dt}{\int_0^1 (\mathrm{x}^2(\mathrm{t}) + \mathrm{u}^2(\mathrm{t}))\,\mathrm{dt}}$$
$$x'(t) = u(t),$$
$$x(0) = 0.$$

Solution: By means of method of moments we obtain the following optimization problem:

$$J = \min \sum_{r=0}^{l-1} \gamma((\mathrm{t}_r^{\ 4} - 2t_r^{\ 2} x(t_r) + x^2(t_r))$$
$$+ (\mathrm{t}_{r-1}^{\ 4} - 2t_{r-1}^{\ 2} x(t_{r-1}) + x^2(t_{r-1})))\frac{h}{2}$$
$$\frac{x_r - x_{r-1}}{h} = m_1(t_r), \qquad \forall\, \mathrm{r} = 1,2,\ldots,\mathrm{l},$$
$$\gamma \sum_{r=0}^{l-1} ((\mathrm{x}^2(t_r) + m_2(t_r)) + (\mathrm{x}^2(t_{r-1}) + m_2(t_{r-1})))\frac{h}{2} = 0,$$
$$\begin{pmatrix} 1 & m_1(t_r) \\ m_1(t_r) & m_2(t_r) \end{pmatrix} \geq 0 \qquad \forall r = 0,1,\ldots l,$$
$$x(0) = 0$$

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By choose $h=\frac{1}{10}$, we obtain the value of objective function as zero and the exact value of objective function is equal 0.

CONCLUSION

In this paper we consider a special type of optimal control problem where the objective functional is given by the ratio of two integrals. For obtain the optimal value of objective function we use method of moments. By this method the non-linear and non-convex optimal control problem transform to a convex optimization problem in moment space. Finally by use of GAMS software we solve the optimization problem.

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