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A New Method for Solving the Generalized Interval-Valued Fuzzy Numbers Linear Programming Problems

A. Mahmoodirad[1*], H. Hassasi[1], S. Molla-Alizadeh-Zavardehi[2], M. J. Esfahani[3]

[1]*Department of Mathematics, Masjed-Soleiman Branch, Islamic Azad University, Masjed-Solieman, Iran*
[2]*Department of Industrial Engineering, Islamic Azad University, Masjed-Soleiman Branch, Masjed-Solieman, Iran*
[3]*Department of Industrial Engineering, Islamic Azad University, Naragh Branch, Naragh, Iran*

*Corresponding author's E-mail: alimahmoodirad@yahoo.com

Abstract

In this paper, we concentrate on linear programming problems in which the cost vector, the technological coefficients and the right-hand side are generalized interval-valued trapezoidal fuzzy numbers. To the best of our knowledge, till now there is no method described in the literature to find the optimal solution of the linear programming problems with generalized interval-valued trapezoidal fuzzy numbers. We apply the signed distance for defuzzification of this problem. The crisp problem obtained after the defuzzification can be solved by the linear programming methods. Finally, we give an illustrative example and its numerical solutions.

Original Article

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INTRODUCTION

The fuzzy set theory was, for the first time, introduced by Zadeh [14] and has found extensive applications in various fields. Bellman and Zadeh [2] were the first to consider the application of the fuzzy set theory in solving optimization problems. The fuzzy set theory is a powerful tool to handle imprecise data and fuzzy expressions that are more natural for humans than rigid mathematical rules and equations. It is obvious that much knowledge in real world situations is fuzzy rather than precise. This theory is being applied extremely in many fields these days. One of these is linear programming problems. The linear programming problem is one of the most frequently applied operation research techniques. Although it has been investigated and expanded for more than six decades by many researchers and from the various point of views, it is still useful to develop new approaches in order to better fit the real world problems within the framework of linear programming [1].

In conventional approach, parameters of linear programming problems must be well defined and precise. However, in real world environment, this is not a realistic assumption. In such cases, using imprecise data such as interval numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, generalized trapezoidal fuzzy numbers (GTFNs) and generalized interval-valued trapezoidal fuzzy numbers (GIVTFNs) for modeling the problem is quite appropriate. In this paper, we focus on GTFNs and GIVTFNs.

The concept of interval-valued fuzzy sets is initially proposed by Gorzalczany [8] and Turksen [10]. Then based on this achievement, Wang and Li [11] defined the expansion operation of interval-valued fuzzy numbers, and proposed the concept and properties of similarity coefficient based on the interval-valued fuzzy numbers. Hong and Lee [9], Wang and Li [13], proposed the distance of interval-valued fuzzy numbers. Chen and Chen [4] presented the concept of GIVTFN based on the concepts of generalized trapezoidal fuzzy number proposed by

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Chen [5] and interval-valued fuzzy number proposed by Wang and Li [11], and proposed a new method for ranking the GIVTFNs.

Obviously, the GIVTFN is more general form of fuzzy numbers, almost all fuzzy numbers can be viewed as its special case, for example, trapezoidal fuzzy number, generalized trapezoidal fuzzy number, interval-valued triangular fuzzy, etc.

For the linear programming problem that the cost vector, the technological coefficients and the right-hand side is GIVTFNs, this paper proposes a new method to obtain the optimal solution. It applies the signed distance for defuzzification of this problem. So, the crisp problem obtained after the defuzzification may be solved by the linear programming methods.

The rest of this paper is organized as follows: In Section 2, we review the basic definitions and concepts the generalized trapezoidal fuzzy numbers and GIVFNs. Section 3 gives the definition of Linear programming with GIVFNs problem. We give a new method for solving linear programming problem with GIVFNs in Section 4. A numerical example is given in section 5. The conclusions are discussed in Section 6.

MATERIALS AND METHODS

Preliminaries

In this section, we briefly review some of the concept of generalized trapezoidal fuzzy numbers. They are crucial for the remainder of this paper.

Definition 1: A fuzzy set $\tilde{A}$, defined on $\mathbb{R}$, is said to be generalized fuzzy number if the following conditions hold:

(a) Its membership function is piecewise continuous function. (b) There exist two intervals [a, b] and [c, d] such that $\tilde{A}$ is strictly increasing on [a, b] and strictly decreasing on [c, d]. (c) $\tilde{A}(x) = w$, For all $x \in [b, c]$, where $0 < w \leq 1$.

Definition 2: A fuzzy number $\tilde{A}$ = (a, b, c, d; w) is said to be a generalized trapezoidal fuzzy number (as shown in Figure 1) if its membership function is given by

$$\tilde{A}(x) = \begin{cases} w\dfrac{x-a}{b-a}, & a \leq x < b \\ w, & b \leq x \leq c \\ w\dfrac{x-a}{b-a}, & c < x \leq d \\ 0, & \text{otherwise} \end{cases}$$

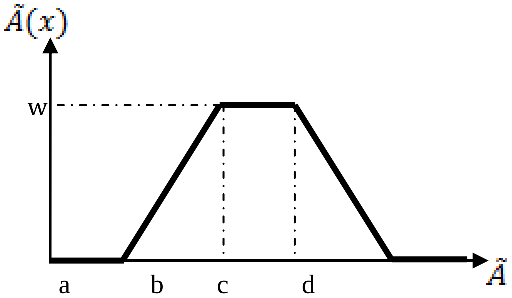

Figure 1. Generalized trapezoidal fuzzy numbers

The elements of the generalized trapezoidal fuzzy numbers are real numbers. If $-1 \leq a \leq b \leq c \leq d \leq 1$, then $\tilde{A}$ is called the normalized trapezoidal fuzzy number. Especially, if w = 1, then the generalized trapezoidal fuzzy number $\tilde{A}$ is called a trapezoidal fuzzy number and denoted as $\tilde{A} = (a, b, c, d)$. If $a < b = c < d$, then $\tilde{A}$ is reduced to a triangular fuzzy number. If $a = b = c = d$, then $\tilde{A}$ is reduced to a real number.

Now, we review arithmetic operations on generalized trapezoidal fuzzy numbers.

Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)$ be two generalized trapezoidal fuzzy numbers. Define,

1. $\tilde{A} \oplus \tilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; w)$,

Where w= minimum $\{w_1, w_2\}$,

2. $\tilde{A} \ominus \tilde{B} = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2; w)$,

where w= minimum $\{w_1, w_2\}$,

3. $\theta\tilde{A} = \begin{cases} (\theta a_1, \theta b_1, \theta c_1, \theta d_1; w_1)\theta > 0, \\ (\theta d_1, \theta c_1, \theta b_1, \theta a_1; w_2)\theta < 0. \end{cases}$

The Generalized Interval-valued trapezoidal fuzzy numbers

In this section, we review some of the concept, notations and arithmetic operations generalized interval-valued trapezoidal fuzzy numbers.

Definition 3 [4,6]: Let $\tilde{A}^L$ and $\tilde{A}^U$ be two generalized trapezoidal fuzzy numbers. A GIVTFN (as shown in Fig. 2) represented by the following:

$\tilde{\tilde{A}} = [\tilde{A}^L, \tilde{A}^U] = [(a_1^l, a_2^l, a_3^l, a_4^l; w^l), (a_1^u, a_2^u, a_3^u, a_4^u; w^u)]$,

Where,

$0 \leq a_1^l \leq a_2^l \leq a_3^l \leq a_4^l \leq 1, 0 \leq a_1^u \leq a_2^u \leq a_3^u \leq a_4^u \leq 1$, $0 \leq w^l \leq w^u \leq 1$ and $\tilde{A}^L \subset \tilde{A}^U$.

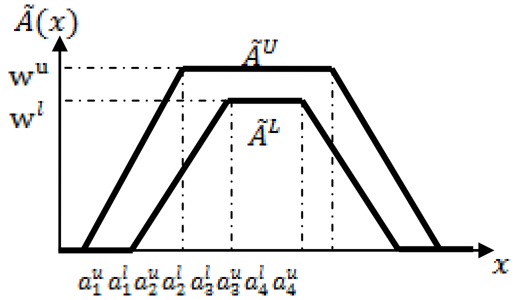

Figure 2. Generalized interval-valued trapezoidal fuzzy numbers

From Figure 2, we can see that a generalized interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$ consist of the lower generalized trapezoidal fuzzy number $\tilde{A}^L$ and the upper generalized trapezoidal fuzzy number $\tilde{A}^U$.

Arithmetic operations

The following we review arithmetic operations between two GIVTFNs.

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Let

$\tilde{\tilde{A}} = [\tilde{A}^L, \tilde{A}^U] = [(a_1^l, a_2^l, a_3^l, a_4^l; w^l), (a_1^u, a_2^u, a_3^u, a_4^u; w^u)]$,

where

$0 \le a_1^l \le a_2^l \le a_3^l \le a_4^l \le 1,\ 0 \le a_1^u \le a_2^u \le a_3^u \le a_4^u \le 1, 0 \le w^l \le w^u \le 1,$

and,

$\tilde{\tilde{B}} = [\tilde{B}^L, \tilde{B}^U] = [(b_1^l, b_2^l, b_3^l, b_4^l; v^l), (b_1^u, b_2^u, b_3^u, b_4^u; v^u)]$

where

$0 \le b_1^l \le b_2^l \le b_3^l \le b_4^l \le 1,\ 0 \le b_1^u \le b_2^u \le b_3^u \le b_4^u \le 1, 0 \le v^l \le v^u \le 1$

, be two GIVTFNs, then, the operations are shown as follows [20, 21].

$$\tilde{\tilde{A}} \oplus \tilde{\tilde{B}} = [(a_1^l + b_1^l, a_2^l + b_2^l, a_3^l + b_3^l, a_4^l + b_4^l; \min(w^l, v^l)), (a_1^u + b_1^u, a_2^u + b_2^u, a_3^u + b_3^u, a_4^u + b_4^u; \min(w^u, v^u))]$$

$$\tilde{\tilde{A}} \ominus \tilde{\tilde{B}} = [(a_1^l - b_4^l, a_2^l - b_3^l, a_3^l - b_2^l, a_4^l - b_1^l; \min(w^l, v^l)), (a_1^u - b_4^u, a_2^u - b_3^u, a_3^u - b_2^u, a_4^u - b_1^u; \min(w^u, v^u))]$$

$$\lambda\tilde{\tilde{A}} = [(\lambda a_1^l, \lambda a_2^l, \lambda a_3^l, \lambda a_4^l; w^l), (\lambda a_1^u, \lambda a_2^u, \lambda a_3^u, \lambda a_4^u; w^u)], \qquad \lambda > 0,$$

$$\lambda\tilde{\tilde{A}} = [(\lambda a_4^l, \lambda a_3^l, \lambda a_2^l, \lambda a_1^l; w^l), (\lambda a_4^u, \lambda a_3^u, \lambda a_2^u, \lambda a_1^u; w^u)], \qquad \lambda < 0,$$

$$\lambda\tilde{\tilde{A}} = [(0,0,0,0; w^l), (0,0,0,0; w^u)], \qquad \lambda = 0.$$

The distance of generalized interval-valued trapezoidal fuzzy numbers

Let $\tilde{\tilde{A}} = [(a_1^l, a_2^l, a_3^l, a_4^l; w^l), (a_1^u, a_2^u, a_3^u, a_4^u; w^u)]$,

be a GIVTFN, the signed distance of $\tilde{\tilde{A}}$ from $\tilde{\tilde{1}}_1$(y-axis at x=1) are as follows [6]:

$$d\left(\tilde{\tilde{A}}, \tilde{\tilde{1}}_1\right) = \frac{1}{8}(a_1^l + a_2^l + a_3^l + a_4^l + 4a_1^u + 2a_2^u + 2a_3^u + 4a_4^u + 3(a_2^u + a_3^u - a_1^u - a_4^u)\frac{w^l}{w^u} - 16).$$

From this, we can conclude that signed distance $d\left(\tilde{\tilde{A}}, \tilde{\tilde{1}}_1\right) \le 0$, because the values of the GIVTFN $\tilde{\tilde{A}}$ are between zero and one.

Definition 4 [6] Let $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ be two GIVTFNs. The ranking of $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ by the signed distances $d\left(\tilde{\tilde{A}}, \tilde{\tilde{1}}_1\right)$ and $d\left(\tilde{\tilde{B}}, \tilde{\tilde{1}}_1\right)$ can be defined as follows:

$d\left(\tilde{\tilde{A}}, \tilde{\tilde{1}}_1\right) > d\left(\tilde{\tilde{B}}, \tilde{\tilde{1}}_1\right)$ iff $\tilde{\tilde{A}} \succ \tilde{\tilde{B}}$,

$d\left(\tilde{\tilde{A}}, \tilde{\tilde{1}}_1\right) = d\left(\tilde{\tilde{B}}, \tilde{\tilde{1}}_1\right)$ iff $\tilde{\tilde{A}} \sim \tilde{\tilde{B}}$,

$d\left(\tilde{\tilde{A}}, \tilde{\tilde{1}}_1\right) < d\left(\tilde{\tilde{B}}, \tilde{\tilde{1}}_1\right)$ iff $\tilde{\tilde{A}} \prec \tilde{\tilde{B}}$.

Linear programming with generalized interval-valued trapezoidal fuzzy numbers

In conventional linear programming problems of the parameters must be well defined and precise. But, in many applications of linear programming problems this assumption is not true. To deal with such situations, the parameters of linear programming problems may be represented as GIVTFNs. Here, we let all parameters of the linear programming problem are GIVTFNs, while the decision variables are crisp. So, a linear programming problem with GIVTFNs is defined as:

Max $\quad \tilde{\tilde{Z}} \approx \tilde{\tilde{C}}X$

s.t. $\quad \tilde{\tilde{A}}X \approx \tilde{\tilde{B}}$ (1)

$X \ge 0.$

Where $\tilde{\tilde{C}} = [\tilde{C}^L, \tilde{C}^U]$, $\tilde{\tilde{A}} = [\tilde{A}^L, \tilde{A}^U]$, and $\tilde{\tilde{B}} = [\tilde{B}^L, \tilde{B}^U]$ are GIVTFNs.

Definition 5: We say that a vector $X \in \mathbb{R}^n$ is a feasible solution to (1) if it satisfies the following conditions:

i) $\tilde{\tilde{A}}X \approx \tilde{\tilde{B}}$, ii) $X \ge 0$.

Definition 6: Let vector $X^* \in \mathbb{R}^n$ be a feasible solution to (1). It is an optimal solution to (1), if for all feasible solution X for (1), we have $\tilde{\tilde{C}}X^* \succcurlyeq \tilde{\tilde{C}}X$.

Definition 7: Let X_1 be an optimal solution to (1). If there exist a vector X_2 such that,

i) $X_2 \ge 0$, ii) $\tilde{\tilde{A}}X_2 \approx \tilde{\tilde{B}}$, iii) $\tilde{\tilde{C}}X_1 \approx \tilde{\tilde{C}}X_2$,

Then, X_2, is said to be an alternative optimal solution for (1).

Proposed method to find the optimal solution

In this section, a new method is proposed to obtain the optimal solution of problem (1), in which all the parameters are represented by GIVTFNs, while the decision variables are crisp. So, the various steps to find the optimal solution of problem (1) are as follows:

Step 1. Substituting $\tilde{\tilde{C}} = [\tilde{C}_j^L, \tilde{C}_j^U]$, $\tilde{\tilde{A}} = [\tilde{A}_{ij}^L, \tilde{A}_{ij}^U]$, and $\tilde{\tilde{B}} = [\tilde{B}_i^L, \tilde{B}_i^U]$ of the problem (1) may be written as:

Max $\quad \sum_{j=1}^n [\tilde{C}_j^L, \tilde{C}_j^U] x_j$

s.t. $\quad \sum_{j=1}^n [\tilde{A}_{ij}^L, \tilde{A}_{ij}^U] x_j = \sum_{i=1}^m [\tilde{B}_i^L, \tilde{B}_i^U]$ *i=1,2,…,m* (2)

$x_j \ge 0$, *j=1,2,…,n*.

Step 2. Applying signed distance d to problem (2), obtained in step 1, we have:

Max $\quad d(\sum_{j=1}^n [\tilde{C}_j^L, \tilde{C}_j^U] x_j)$

s.t. $\quad (\sum_{j=1}^n [\tilde{A}_{ij}^L, \tilde{A}_{ij}^U] x_j) = d(\sum_{i=1}^m [\tilde{B}_i^L, \tilde{B}_i^U])$ *i=1,2,…,m*(3)

$x_j \ge 0$, *j=1,2,…,n*.

Using arithmetic operations, defined in subsection 2.2.2,and the signed distance, stated in section 2.2.3, the problem (3) is converted into an optimization problem where may be solve by the linear programming methods.

Step 3. Find the optimal solution x_j by solving the problem obtained in step 2.

Step 4. Find the optimal value by putting the values of x_j in the objective function of the problem (2) where it is a generalized interval-valued trapezoidal fuzzy number.

Numerical example

Example: Consider the following applied problem. A Company manufactures two kinds of products (I), (II) with an uncertain profit, $\tilde{\tilde{C}}_1, \tilde{\tilde{C}}_1$ Dollar per unit respectively. If objective of this company is to maximize the profit in return to the total cost, provided that the company has a raw materials for manufacturing and suppose the material

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needed per ponds are $\tilde{\tilde{A}}_{11}$, $\tilde{\tilde{A}}_{12}$ and the supply for this raw material is restricted to $\tilde{\tilde{B}}_1$ pounds, it is also assumed that twice of production of (II) is $\tilde{\tilde{B}}_2$. In this case, if we consider, x_1, x_2 to be the amount of units of (I), (II) to produce then the above problem can be formulated as:

$$
\begin{aligned}
&\text{Max} \quad \tilde{\tilde{Z}} \approx \tilde{\tilde{C}}_1 x_1 \oplus \tilde{\tilde{C}}_2 x_2 \\
&\text{s.t.} \quad \tilde{\tilde{A}}_{11} x_1 \oplus \tilde{\tilde{A}}_{12} x_2 \approx \tilde{\tilde{B}}_1, \qquad (4) \\
&\quad . \quad \tilde{\tilde{A}}_{21} x_1 \oplus \tilde{\tilde{A}}_{22} x_2 \approx \tilde{\tilde{B}}_2, \\
&x_1, x_2 \geq 0.
\end{aligned}
$$

Where the values of $\tilde{\tilde{C}}_1, \tilde{\tilde{C}}_2, \tilde{\tilde{A}}_{11}, \tilde{\tilde{A}}_{12}, \tilde{\tilde{A}}_{21}, \tilde{\tilde{A}}_{22}, \tilde{\tilde{B}}_1, \tilde{\tilde{B}}_2$ are:

$$
\begin{aligned}
\tilde{\tilde{C}}_1 &= [(0.2, 0.5, 0.7, 0.9; 0.3), (0.1, 0.4, 0.8, 0.95; 0.5)], \\
\tilde{\tilde{C}}_2 &= [(0.5, 0.7, 0.9, 1; 0.2), (0, 0.6, 0.95, 1; 0.4)], \\
\tilde{\tilde{A}}_{11} &= [(0.3, 0.7, 0.85, 0.9; 0.4), (0.2, 0.5, 0.9, 1; 0.6)], \\
\tilde{\tilde{A}}_{12} &= [(0.1, 0.3, 0.5, 0.7; 0.8), (0, 0.2, 0.6, 0.9; 0.9)], \\
\tilde{\tilde{A}}_{12} &= [(0.5, 0.6, 0.7, 0.8; 0.3), (0.1, 0.4, 0.9, 1; 0.7)], \\
\tilde{\tilde{A}}_{22} &= [(0.3, 0.7, 0.8, 0.95; 0.4), (0.1, 0.5, 0.9, 1; 0.8)], \\
\tilde{\tilde{B}}_1 &= [(0.4, 0.5, 0.7, 0.8; 0.5), (0, 0.2, 0.8, 0.9; 0.7)], \\
\tilde{\tilde{B}}_2 &= [(0.5, 0.6, 0.7, 0.8; 0.4), (0.1, 0.55, 0.75, 0.85; 0.9)],
\end{aligned}
$$

Solution: Using Step 1 and Step 2 of proposed method, the following problem is obtained for problem (4):

$$
\begin{aligned}
&\text{Max} \quad z = 1.1406\, x_1 + 1.3781\, x_2 - 2 \\
&\text{s.t.} \quad 1.3438\, x_1 + 0.85\, x_2 = 1.0268, \quad (5) \\
&1.4175\, x_1 + 1.292\, x_2 = 1.1833, \\
&x_1, x_2 \geq 0.
\end{aligned}
$$

By omitting fix number, that is -2, from the objective function of problem (5), we have:

$$
\begin{aligned}
&\text{Max} \quad z_1 = 1.1406\, x_1 + 1.3781\, x_2 \\
&\text{s.t.} \quad 1.3438\, x_1 + 0.85\, x_2 = 1.0268, \qquad (6) \\
&1.4175\, x_1 + 1.292\, x_2 = 1.1833, \\
&x_1, x_2 \geq 0.
\end{aligned}
$$

The problem (6) is a linear programming problem and its optimal solution is: $x_1^* = 0.6037$, $x_2^* = 0.2534$, $z_1{}^* = 1.0379$. So, the optimal value for problem (5) is as follows: $z^* = 1.0379 - 2 = -0.9621$.

The optimal value of objective function for problem (4) is a generalized interval-valued trapezoidal fuzzy number as follows:

$$
\tilde{\tilde{Z}}^* \approx \tilde{\tilde{C}}_1 x_1^* \oplus \tilde{\tilde{C}}_2 x_2^* \approx \begin{bmatrix} (0.1207, 0.3018, 0.4226, 0.5433; 0.3), \\ (0.0604, 0.2415, 0.483, 0.5735; 0.5) \end{bmatrix} \oplus \begin{bmatrix} (0.1267, 0.1774, 0.2281, 0.2534; 0.2), \\ (0, 0.152, 0.2441, 0.2534; 0.4) \end{bmatrix}
$$

$$
= [(0.2474, 0.4792, 0.6507, 0.7967; 0.2), (0.6004, 3935, 0.7271, 0.8269)].
$$

Therefore we have $d\left(\tilde{\tilde{Z}}^*, \tilde{\tilde{1}}_1\right) = -0.8714 \leq 0.$

CONCLUSIONS

This study, has presented a new method to find the fuzzy optimal solution of linear programming problem in which, the cost vector, the technological coefficients and the right-hand side are GIVTFNs. Then, we applied the signed distance for defuzzification of this problem. So, the crisp problem obtained after the defuzzification is solved by the linear programming methods. By using the proposed method the optimal solution of problems with GIVTFNs coefficients, occurring in real life problems, can be easilyobtained. This method can be applied for solving transportation problems with GIVTFNs as transportation cost or supply and demand values.

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